Property of Time-Invariance (Shift-Invariance) for a System Under Observation

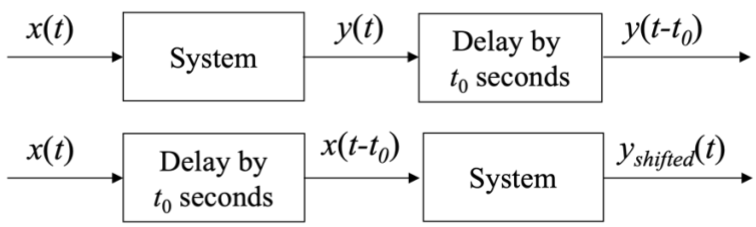
Prof. Brian L. Evans and Mr. Jaehong Moon
The University of Texas at Austin, bevans@ece.utexas.edu

September 20, 2018 and updated January 2, 2023, and January 23, 2023

When evaluating system properties, we treat a system as a closed box and analyze the relationships between input signals and their corresponding output signals. This process assumes that after inputting a signal, we can return the system to its original state.

A continuous-time system with input signal $x(t)$ and output signal $y(t)$ is time-invariant (shift-invariant) if whenever the input signal is delayed by t_0 seconds, then the output signal will always be delayed by t_0 seconds as well for all real values of t_0.

A way to visualize the time-invariance property is to show the equivalence between



That is, does $y_{shifted}(t) = y(t - t_0)$ for all possible real constant values of t_0?

One-sided infinite observation. Let's consider the system under observation for $t \geq 0^-$. Time 0^- means a time of 0 seconds before occurrence of a Dirac delta occurring at the origin. We can only observe $x(t)$ for $t \geq 0^-$ and $y(t)$ for $t \geq 0^-$. This means that we can only observe $x(t - t_0)$ for $t \geq t_0$ and $y(t - t_0)$ for $t \geq t_0$.

Example. Consider a delay system that delays the input by T seconds, and we can only observe the input signal and the output signal for $t \geq 0^-$.

Conceptually, the delay block can be thought as a long wire that conducts electricity from the input to the output. Assuming electrons travel at 2/3 the speed of light, the length of the wire would be (2/3) c T where c is the speed of light (3 x 10^8 m/s). Such an implementation would be impractical, but nonetheless helpful in analyzing the system.

The first observed output value $y(0)$ would be due to the initial conditions in the delay system. In fact, the first T seconds of the output would due solely to the initial conditions in the system. For input $x(t)$ and output $y(t)$, once the initial conditions have been output, $y(T) = x(0)$. That is, it takes T seconds for an input value (voltage) to arrive at the output.

The initial conditions for the delay system consist of the voltage values at different points in the wire at $t = 0$. Let's denote these voltage values $v(t)$ for $-T < t \leq 0$. That is, $v(0)$ will be first, and $v(-T)$ will be the last, value among the initial conditions to be output. The spatial location for the voltage $v(t)$ for $-T < t \leq 0$ is (-2/3) c t meters from the output location.

Let $x(t) = 0$ for $0 \leq t < T$ and 1 for $t \geq T$. For input $x(t)$, the output is

$$y(t) = \begin{cases} v(-t) & \text{for } 0 \leq t < T \\ x(t-T) & \text{for } t \geq T \end{cases} = \begin{cases} v(-t) & \text{for } 0 \leq t < T \\ 0 & \text{for } T \leq t < 2T \\ 1 & \text{for } t \geq 2T \end{cases}$$

Let's keep the same initial conditions, i.e. $v(t)$ for $-T < t \leq 0$, and the same definition for signal $x(t)$. Now, we input $x(t-t_0)$ into the delay system

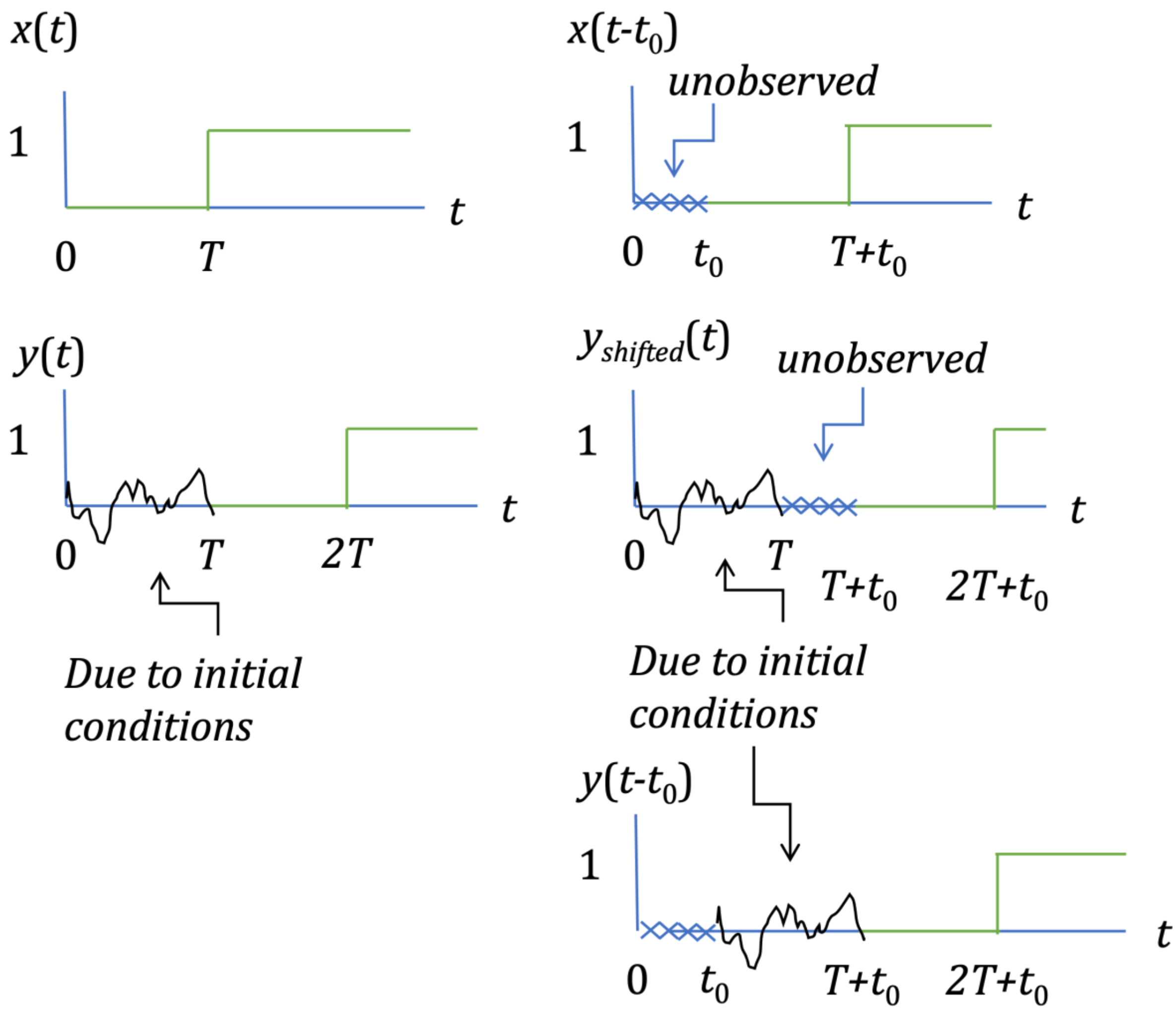
$$x(t-t_0) = \begin{cases} unobserved & \text{for } 0 \leq t < t_0 \\ 0 & \text{for } t_0 \leq t < T + t_0 \\ 1 & \text{for } t \geq T + t_0 \end{cases}$$

and the output is

$$y_{shifted}(t) = \begin{cases} v(-t) & \text{for } 0 \leq t < T \\ unobserved & \text{for } T \leq t < T + t_0 \\ 0 & \text{for } T + t_0 \leq t < 2T + t_0 \\ 1 & \text{for } t \geq 2\,T + t_0 \end{cases} = \begin{cases} v(-t) & \text{for } 0 \leq t < T \\ unobserved & \text{for } T \leq t < T + t_0 \\ y(t-t_0) & \text{for } t \geq T + t_0 \end{cases}$$

$y_{shifted}(t)$ only equals $y(t-t_0)$ for $t \geq T + t_0$ because the initial conditions did not shift in time even though the input did.

Plots of the signals are given next followed by an analysis of initial conditions:



If the system were time-invariant, then $y_{shift}(t) = y(t-t_0)$ for all real t_0 and $t \geq 0^-$. This holds for for $t \geq T + t_0$. For $0 \leq t < T + t_0$, all unobserved values and initial conditions would have to be equal to a constant value.